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Thomas-Fermi approximation to functional determinants in external gauge potentials

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The functional determinant and vacuum expectation value of a fermion field in an external gauge potential are evaluated in a Thomas-Fermi approximation. This approximation should be valid for slowly varying potentials. The expressions are readily transcribable to a lattice. We do not encounter the problem of doubling of fermionic degrees of freedom.

I. INTRODUCTION

The inclusion of fermions in lattice calculations has run into two major problems. The first one is calculational in that Monte Carlo integrations over Grassmann variables cannot be performed and one is forced into approximate numerical methods[1] for the evaluation of inverses and determinants of functional matrices; an extreme approximation to the determinant consists of setting it identically equal to one[2] (quenched approximation). The other difficulty is that the naive continuum limit of fermions on a lattice has too many degrees of freedom. This problem has been dealt with through the introduction of a quadratic term into the Lagrangian for the fermions[3] (Wilson fermions) or by appropriately staggering them on the lattice[4] (Kogut-Susskind fermions). The first method yields an action that is not invariant under chiral transformations, even in the massless fermion limit. The second technique reduces the number of fermionic degrees of freedom but does not completely eliminate the multiplication of fermionic species. These two problems could be avoided should we be able to perform the integrations over the fermions in the continuum resulting in a functional of the gauge potentials. These expressions could then be discretized and placed on a lattice. In this article we shall present an approximate evaluation of the functional determinant for fermions moving in an external gauge potential as well as the expectation value of $\langle\bar{\psi}\psi\rangle$. The approximation we shall use is a variant of the Thomas-Fermi method in that we treat the external gauge potentials as slowly varying. Confinement and chiral-symmetry breaking are long-distance problems and thus rapid short-distance variations of the potentials are not expected to play a dominant role. For technical reasons we shall, for the present, limit ourselves to the gauge group SU(2).

The properties of functional determinants for particles of various spins and isospins moving in constant external potentials were previously studied by Brown and Weisberger.[5] Although they did discuss the situation of a spin-$\frac{1}{2}$ color-$\frac{1}{2}$ particle moving in such a potential, these authors did not obtain an expression for the functional determinant in a form suitable for the purposes discussed previously. For both the effective action and for $\langle\bar{\psi}\psi\rangle$ we will obtain renormalized expressions. In a subsequent publication we hope to return to a Monte Carlo calculation using these results.

In Sec. II the Thomas-Fermi approximation for particles moving in external gauge potentials is discussed, with special attention paid to the gauge-invariance properties of this approximation. Conclusions will be given in Sec. III along with some observations on configurations of gauge potentials that may possibly be responsible for chiral-symmetry breakdown. The details of the evaluation of the effective action are presented in the Appendix.

II. THOMAS-FERMI APPROXIMATION

The contribution of a single Dirac fermion of mass m moving in an external gauge potential A to the effective action for the gauge potential is

$$-F=\int d^4x\langle x\mid \operatorname{tr}\ln(i\partial\!\!\!/-A\!\!\!/-m)\mid x\rangle+\text{const}\ . \tag{2.1}$$

We are working in Euclidean space and the coupling constant is absorbed into the definition of the potential. The gauge potentials themselves are matrix valued. The trace is over all spin and color variables. The other quantity we shall be interested in is the expectation value of $\bar{\psi}\psi$. This is related to F by

$$\int d^4x\langle\bar{\psi}(x)\psi(x)\rangle=\partial F/\partial m\ . \tag{2.2}$$

Both of these may be obtained from

$$Z(x;\eta)=\langle x\mid \operatorname{tr}\exp[-\eta(i\partial\!\!\!/-A\!\!\!/-m)]\mid x\rangle \tag{2.3}$$

by using the relations

$$F=\int d^4x\int_0^\infty\frac{d\eta}{\eta}[Z(x;\eta)-\langle x\mid e^{-\eta M}\mid x\rangle]\ , \tag{2.4a}$$

$$\int d^4x\langle\bar{\psi}(x)\psi(x)\rangle=\int d^4x\int_0^\infty d\eta\, Z(x;\eta)\ . \tag{2.4b}$$

The diagonal matrix element of the integrand of Eq. (2.3) is the object we shall try to evaluate in a Thomas-Fermi approximation. Namely, we approximate a diagonal matrix element of a function of the operators P and Q, with $[P,Q]=-i$, by

$$\langle x\mid F(P,Q)\mid x\rangle=\int\frac{dp}{2\pi}F(p,x)\ . \tag{2.5}$$

Had we been dealing with the problem of a particle moving in an ordinary potential, this approximation would have consisted of treating the potential-energy term as a constant c number whose value is the potential energy at x. Gauge invariance prevents us from treating the potential as a constant; applying a gauge transformation changes it. Before developing a variant of this approximation suitable for this problem we will write a different representation for the function $Z(x;\eta)$ of Eq. (2.3). A standard path-integral representation for this function is

$$Z(x;\eta)=\int [dx\,dp] \times P\exp\left[-\int_0^\eta d\tau\left[ip\cdot\frac{dx}{d\tau}-(\not p-\not A-m)\right]\right], \tag{2.6}$$

where we integrate over all paths that begin and end at x. The integrand is path ordered both in the color and Dirac matrices. An equivalent expression is

$$Z(x;\eta)=\int [dx\,dp] \times P\exp\left[-\int_0^\eta d\tau[ip\cdot dx/d\tau-(\not p-m)]\right] \times P\exp\left[i\oint A\cdot dx\right]. \tag{2.7}$$

The first path ordering refers to the Dirac matrices only, while the second refers to the color ones.

Again, because of the choice of gauge freedom we cannot set the potential A equal to its value at the point x. The lowest-order Thomas-Fermi approximation will consist of choosing, for each x, a constant potential that reproduces

$$P\exp\left[i\int A\cdot dx\right]$$

for small paths beginning and ending at x. As discussed in Ref. 5, one can find a constant gauge transformation and Lorentz transformation that brings an SU(2) potential to the form where only three out of the twelve components are nonzero. Thus there are only three combinations of contours that we can reproduce. With

$$M_{\mu\nu}=P\exp\left[i\oint_{C_{\mu\nu}}A\cdot dx\right]-1,$$

where $C_{\mu\nu}$ is the square contour of side a as shown in Fig. 1(a), these combinations are given by

$$G_2=\lim_{a\to 0}\frac{-2}{a^4}\mathrm{tr}\sum_{\mu\nu}M_{\mu\nu},$$
$$G_3=\lim_{a\to 0}\frac{2}{3a^6}\mathrm{tr}M_{\mu\nu}M_{\nu\lambda}M_{\lambda\mu}, \tag{2.8}$$
$$G_4=\lim_{a\to 0}\frac{1}{a^8}\mathrm{tr}(M_{\mu\nu}M_{\mu\nu}M_{\lambda\rho}M_{\lambda\rho}-2M_{\mu\lambda}M_{\nu\lambda}M_{\mu\rho}M_{\nu\rho}).$$

The types of infinitesimal loops around the point x which are combined in these expressions are shown in Fig. 1. For ordinary potential problems higher-order corrections to the Thomas-Fermi approximation[6] retain higher-order derivatives of the potential at each point; adding terms linear, quadratic, etc., in x to the gauge potential would permit us to reproduce the path-ordered integrals for a larger class of loops. The results for the loops presented in Fig. 1 may also be given in terms of the field strengths $F_{\mu\nu}$:

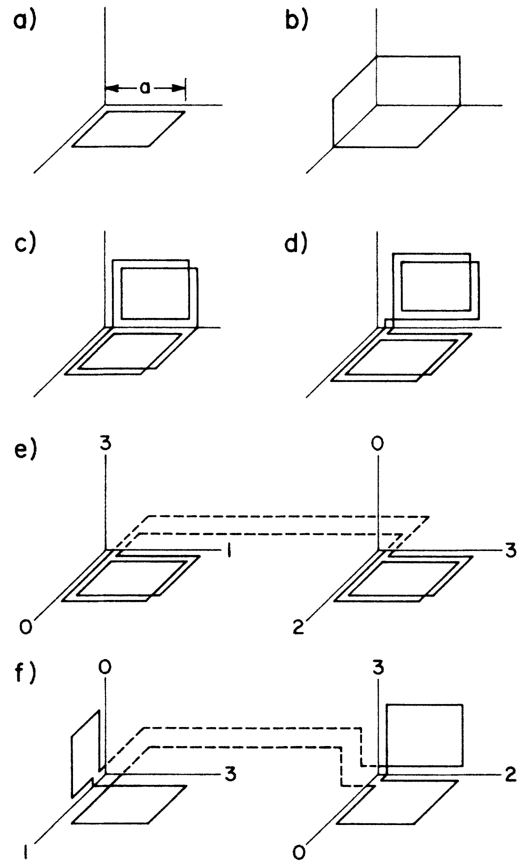

FIG. 1. Types of contours which contribute to G_2, G_3, and G_4. G_2 involves contours of type (a). G_3 involves type (b). G_4 involves types (c)—(f). A variety of other contours are involved in the definitions of the G's which cancel parts that are infinite as a goes to zero.

$$G_2=\mathrm{tr}F_{\mu\nu}F_{\mu\nu},$$
$$G_3=-\frac{2i}{3}\mathrm{tr}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}, \tag{2.9}$$
$$G_4=\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho}-2F_{\mu\lambda}F_{\nu\lambda}F_{\mu\rho}F_{\nu\rho}).$$

For non-Abelian theories the field strengths do not determine the field configurations;[7] again, only larger loops will have the property of not being expressible in terms of the $F_{\mu\nu}$'s. For the present case the Thomas-Fermi approximation becomes

$$Z(x;\eta)=\int d^4p\,\mathrm{tr}\exp[-\eta(\not p-\not A-m)]. \tag{2.10}$$

Answers will be presented using $\partial^2F/(\partial m^2)^2$,

$$F=\int_{m^2}^\infty dm'^2(m'^2-m^2)\left[\frac{\partial^2F}{(\partial m'^2)^2}-\mathrm{tr}\frac{F_{\mu\nu}F_{\mu\nu}}{48\pi^2m'^4}\right] + \frac{1}{48\pi^2}\mathrm{tr}F_{\mu\nu}F_{\mu\nu}[\ln(\Lambda^2/m^2)-1]. \tag{2.11}$$

A term behaving as $F_{\mu\nu}F_{\mu\nu}/m^2$ must be subtracted prior to the m'^2 integration and then added on as a logarithmically infinite fermion contribution to the coupling-constant renormalization. Details of the evaluation are presented in the Appendix. Using the notation

$$C(\tau)=\left[\tau^3+4\tau^2\left[\frac{1}{4}\frac{G_4}{G_3}+m^2\right]-\tau^2G_2-4\tau m^2G_3\right]^{1/2},$$
$$z_1(\tau)=2\tau\left[\frac{1}{4}\frac{G_4}{G_3}+m^2\right]+C(\tau)\,, \tag{2.12}$$

we find

$$\partial^2F/(\partial m^2)^2=\frac{1}{8\pi^2}\int d^4x\int_0^\infty d\tau\sqrt{\tau}\,\mathrm{Re}\left[\left[\frac{[z_1(\tau)-G_3]^3}{z_1{}^3(\tau)-\tau z_1{}^2(\tau)\dfrac{G_4}{G_3}+\tau^2z_1(\tau)G_2-\tau^3G_3}\right]^{1/2}C(\tau)^{-1}-\frac{1}{\tau(4m^4+\tau)^{1/2}}\right]. \tag{2.13}$$

These are the main results of this work. We may immediately transcribe these results to a lattice version of this problem by regarding the curves of Fig. 1 as going around elementary plaquettes.

III. CONCLUSIONS

Equations (2.11)—(2.13) provide us with an expression for the fermion-loop correction to the free effective action of an SU(2) gauge field theory and for the fermion propagator. These expressions may be put into a version suitable for a lattice calculation without encountering any problems of doubling of fermion degrees of freedom.

Equation (2.13) is singular for vanishing masses and for potential configurations in which G_3 is small while G_2 and G_4 remain finite. Although we cannot present any rigorous results we speculate that these are the configurations that are, in this approximation, responsible for chiral-symmetry breakdown. These matters should be settled by numerical calculations.

ACKNOWLEDGMENT

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APPENDIX

We wish to further evaluate the approximate expression for F given by (2.4a) and (2.10). The η integration can be redone to give

$$-F=\int\frac{d^4x\,d^4p}{(2\pi)^4}\mathrm{tr}\ln\left[\frac{\not p-\not A(x)-m}{M}\right]=\int\frac{d^4x\,d^4p}{(2\pi)^4}\ln\left[\det\left[\frac{\not p-\not A(x)-m}{M}\right]\right]. \tag{A1}$$

The determinant was worked out in Ref. 5 in a gauge and Lorentz frame in which $A_0=0$, $A_i^a=A_i\delta_i^a$, $i=1$—3 is the spatial index and a is color. Defining

$$F=\int d^4x\,f(x)$$

and using the integral representation for the logarithm gives

$$f(x)=\int\frac{d^4p}{(2\pi)^4}\int_0^\infty\frac{d\eta}{\eta}(\exp\{-\eta[H^2+(p_0{}^2+m^2)A_1{}^2A_2{}^2A_3{}^2]\}-e^{-\eta M^8})\,, \tag{A2}$$

where

$$H=(p^2+\tfrac{1}{4}A^2+m^2)^2-\sum_{i=1}^{3}p_i{}^2A_i{}^2-\tfrac{1}{4}\sum_{i<j}A_i{}^2A_j{}^2\,. \tag{A3}$$

Perturbatively (A1) has infinite zeroth-, second-, and fourth-order terms. They can be subtracted from our expressions for f, and we will do so. For simplicity we will not include them in our expressions until the end, but we will invoke them when convenient. In particular, we choose to drop the constant term $e^{-\eta M^8}$ in (A2).

Using the formula

$$\lim_{\epsilon_1\to0}\left[\frac{\eta}{4\pi}\right]^{1/2}\int_{-\infty}^{\infty}dy\exp\left[-\eta\left[\frac{y^2}{4}-iH(y+i\epsilon_1)\right]\right]=e^{-\eta H^2}$$

gives

$$f(x)=\int_0^\infty \frac{d\eta}{\sqrt{\eta}}\int\frac{d^4p}{(2\pi)^4\sqrt{4\pi}}\int dy\,\exp\left[-\eta\left[\frac{y^2}{4}-iH(y+i\epsilon_1)+(p_0{}^2+m^2)A_1{}^2A_2{}^2A_3{}^2\right]\right]. \tag{A4}$$

Using the formula

$$\lim_{\epsilon_2\to 0}\left[\frac{\eta}{4\pi(\epsilon_1-iy)}\right]^{1/2}\int_{-\infty}^{\infty}dz\,\exp\left[-\eta\left[\frac{z^2}{4(\epsilon_1-iy)}-i(z+i\epsilon_2)(p^2+\tfrac{1}{4}A^2+m^2)\right]\right]=\exp[i\eta(y+i\epsilon_1)(p^2+\tfrac{1}{4}A^2+m^2)^2]$$

gives

$$f(x)=\frac{1}{4\pi}\int dy\int dz\int d\eta\int\frac{d^4p}{(2\pi)^4}\frac{1}{(\epsilon_1-iy)^{1/2}}$$
$$\times\exp\left[-\eta\left[\frac{y^2}{4}+\frac{z^2}{4(\epsilon_1-iy)}+i(y+i\epsilon_1)\sum_{i=1}^{3}p_i{}^2A_i{}^2+\frac{i}{4}(y+i\epsilon_1)\sum_{i<j}A_i{}^2A_j{}^2\right.\right.$$
$$\left.\left.-i(z+i\epsilon_2)(p^2+\tfrac{1}{4}A^2+i\epsilon_2)+(p_0{}^2+m^2)A_1{}^2A_2{}^2A_3{}^2\right]\right]. \tag{A5}$$

The ϵ's were introduced to make the p integration manifestly finite. They will also determine how the integration contours pass around branch points.

The spatial part of the momentum appears in the exponential as

$$\sum_{i=1}^{3}p_i{}^2[i\eta(z+i\epsilon_2)-i\eta(y+i\epsilon_1)A_i{}^2].$$

For the p integral to converge, we therefore need

$$\epsilon_2>\epsilon_1A_i{}^2 \text{ for all } i. \tag{A6}$$

The p integrations can now be done, giving

$$f(x)=\frac{1}{64\pi^3}\int\frac{dy}{(\epsilon_1-iy)^{1/2}}\int dz\int_0^\infty\frac{d\eta}{\eta^2}Re^{-\eta P}, \tag{A7}$$

where

$$R=[A_1{}^2A_2{}^2A_3{}^2-i(z+i\epsilon_2)]^{1/2}\times\prod_{j=1}^{3}[i(y+i\epsilon_1)A_j{}^2-i(z+i\epsilon_2)]^{-1/2} \tag{A8}$$

and

$$P=\frac{y^2}{4}+\frac{z^2}{4(\epsilon_1-iy)}+\frac{i}{4}(y+i\epsilon_1)\sum_{i<j}A_i{}^2A_j{}^2-i(z+i\epsilon_2)(\tfrac{1}{4}A^2+m^2)+m^2A_1{}^2A_2{}^2A_3{}^2. \tag{A9}$$

The η integration diverges at the lower bound. Cutting it off at η_0 and rescaling $\eta=\eta_0u$ gives

$$f=\lim_{\eta_0\to 0}\frac{1}{64\pi^3}\int\frac{dy}{(\epsilon_1-iy)^{1/2}}\int dz\frac{1}{\eta_0}\int_1^\infty\frac{du}{u^2}Re^{-u\eta_0P}$$
$$=\lim_{\eta_0\to 0}\frac{1}{64\pi^3}\int\frac{dy}{(\epsilon_1-iy)^{1/2}}\int dz\frac{1}{\eta_0}RE_2(\eta_0P). \tag{A10}$$

For small z, $E_2(z)$ is given approximately by[8]

$$E_2(z)\cong z\ln z+z(\gamma-1)+1+O(z^2)$$

so

$$f=\lim_{\eta_0\to 0}\frac{1}{64\pi^3}\int\frac{dy}{(\epsilon_1-iy)^{1/2}}\times\int dz\left[\frac{1}{\eta_0}R-RP[1-\gamma-\ln(\eta_0M^8)]+RP\ln\frac{P}{M^8}\right]. \tag{A11}$$

In the z plane, R has branch points at $z=-iA_1{}^2A_2{}^2A_3{}^2-i\epsilon_2$ and $z=yA_j{}^2+i\epsilon_1A_j{}^2-i\epsilon_2$, which are both in the lower half-plane because of (A6). Since R goes as z^{-2} for large z, we can do the z integration of the first term in (A11), giving zero.

The z integral of RP does not converge. The branch points of RP are again in the lower half-plane, so we can integrate to a cutoff radius r_z and close the contour with a semicircle of radius r_z in the upper half-plane giving zero, or

$$\int_{-r_z}^{r_z}dz\,R(z)P(z)=-ir_z\int_0^\pi d\theta\,e^{i\theta}R(r_ze^{i\theta})P(r_ze^{i\theta}).$$

Expanding this in powers of r_z gives

$$-i\int_0^\pi d\theta\left[-\frac{r_ze^{i\theta}}{4(\epsilon_i-iy)}+\frac{4\epsilon_2+A_1{}^2A_2{}^2A_3{}^2}{8(\epsilon_1-iy)}+i(\tfrac{1}{8}A^2+m^2)\right]=-\frac{r_z}{2(\epsilon_1-iy)}-i\pi\frac{4\epsilon_2+A_1{}^2A_2{}^2A_3{}^2}{8(\epsilon_1-iy)}+\pi(\tfrac{1}{8}A^2+m^2).$$

The first term is constant in the A's, so it is removed by the subtraction of the zeroth-order term. The y integration of the second term gives zero. The third is removed by the zeroth- and second-order perturbative subtractions.

η_0 can now be taken to zero, and we are left with

$$f=\frac{1}{64\pi^3}\int\frac{dy}{(\epsilon_1-iy)^{1/2}}\int dz\,RP\ln\frac{P}{M^8}. \tag{A12}$$

Using the formula

$$g(x)=g(y)+(x-y)\frac{dg}{dx}(y)+\int_x^y dx'(x'-x)\frac{\partial^2 g}{\partial x^2}(x')$$

for $RP\ln(P/M^8)$ as a function of m^2 gives

$$f=\frac{1}{64\pi^3}\int\frac{dy}{(\epsilon_1-iy)^{1/2}}\int dz\left[RP\ln\left[\frac{\Lambda^2P_1}{M^8}\right]+RP-\Lambda^2RP_1+\int_{m^2}^{\Lambda^2}dm'^2(m'^2-m^2)\frac{\partial^2}{(\partial m'^2)^2}RP(m'^2)\ln\left[\frac{P(m'^2)}{M^8}\right]\right], \tag{A13}$$

where $P=P_0+m^2P_1$, Λ^2 has been assumed large and only terms which do not vanish as Λ^2 goes to infinity have been kept.

The terms RP and $-\Lambda^2RP_1$ have essentially been dealt with above and are removed by the subtractions. The z integral of $RP\ln(\Lambda^2P_1/M^8)$ can be done like that of RP since the branch points are again in the lower half-plane. The leading terms in r_z are

$$-i\int_0^\pi d\theta\left[-\frac{r_ze^{i\theta}}{4(\epsilon_1-iy)}+\frac{4\epsilon_2+A_1{}^2A_2{}^2A_3^2}{8(\epsilon_1-iy)}+i(\tfrac{1}{8}A^2+m^2)\right]\left[\ln\left[\frac{\Lambda^2r_z}{M^8}\right]+i\left[\theta-\frac{\pi}{2}\right]+\frac{i(\epsilon_2+A_1{}^2A_2{}^2A_3{}^2)}{r_z}e^{-i\theta}\right].$$

As before, most of the terms are either subtracted off or are finite as r_z goes to infinity and zero when the y integration is performed. However, the term

$$-i\ln\left[\frac{\Lambda^2r_z}{M^8}\right]\int_0^\pi d\theta\frac{A_1{}^2A_2{}^2A_3{}^2}{8(\epsilon_1-iy)}=-\frac{i\pi A_1{}^2A_2{}^2A_3{}^2}{8(\epsilon_1-iy)}\ln\left[\frac{\Lambda^2r_z}{M^8}\right]$$

gives, after y is integrated to a radius r_y using the same method used for the z integral,

$$\frac{i\pi}{2\sqrt{2}}A_1{}^2A_2{}^2A_3{}^2r_y{}^{-1/2}\ln\left[\frac{\Lambda^2r_z}{M^8}\right]. \tag{A14}$$

This is ill defined.

The terms in (A13) outside of the m'^2 integral cancel the Λ^2-dependent part of this integral. We will show, however, that perturbatively this integral is finite beyond the fourth order. Hence, (A14) must be finite. To have the right units, the sixth-order term must be proportional to m^{-2} in F. (A14) is independent of m, so it must be zero.

Letting Λ^2 go to infinity, we are left with

$$f=\frac{1}{64\pi^3}\int\frac{dy}{(\epsilon_1-iy)^{1/2}}\int dz\int_{m^2}^\infty dm'^2(m'^2-m^2)\times\frac{R(P_1)^2}{P_0+m'^2P_1}. \tag{A15}$$

The z integral converges. P is a quadratic polynomial in z, with roots at

$$z_2^1=2(y+i\epsilon_1)(\tfrac{1}{4}A^2+m^2)\mp iC(iy), \tag{A16}$$

where

$$C(iy)=i\left[4(y+i\epsilon_1)^2(\tfrac{1}{4}A^2+m^2)^2+iy^2(y+i\epsilon_1)-(y+i\epsilon_1)^2\sum_{i<j}A_i{}^2A_j{}^2+4i(y+i\epsilon_1)m^2A_1{}^2A_2{}^2A_3{}^2\right]^{1/2}, \tag{A17}$$

where the factors of i are introduced to make the final form more convenient.

By separating the real and imaginary parts of $-C^2$, it can be shown this quantity goes clockwise around the origin as y goes from $-\infty$ to ∞, crossing the real axis once. Thus, if the phase of $-iC$ is defined to have a positive imaginary part at some real y, it is always positive for real y and so z_1 is always in the upper half z plane.

From (A16) onward we have dropped ϵ_2 since it no longer determines how the integration contours pass around branch points. The z integration contour had to be deformed into the upper half-plane first. Closing it around the pole at z_1 and writing

$$P=\frac{1}{4(\epsilon_1-iy)}(z-z_1)(z-z_2)$$

gives

$$f=\frac{ie^{i\pi/4}}{16\pi^2}\int_{m^2}^\infty dm'^2(m'^2-m^2)\int dy(y+i\epsilon_1)^{1/2}(A_1{}^2A_2{}^2A_3{}^2-iz_1)^{3/2}\prod_{j=1}^3[i(y+i\epsilon_1)A_j{}^2-iz_1]^{-1/2}C(iy)^{-1}. \tag{A18}$$

There is a branch point at $y=-i\epsilon_1$. The others are all a finite distance away from the origin, on the imaginary axis, so for simplicity we will deform the contour to pass above the origin and let ϵ_1 go to zero.

The branch points of C are at $y=0$ and

$$iy^2+4y(\tfrac{1}{4}A^2+m'^2)^2-y\sum_{i<y}A_i{}^2A_j{}^2+4im'^2A_1{}^2A_2{}^2A_3{}^2=0$$

or

$$y=2i(\tfrac{1}{4}A^2+m'^2)^2-\frac{i}{2}\sum_{i<j}A_i^2A_j^2 \pm i\left[\left[2(\tfrac{1}{4}A^2+m'^2)^2-\tfrac{1}{2}\sum_{i<j}A_i^2A_j^2\right]^2 +4m'^2A_1^2A_2^2A_3^2\right]^{1/2} \tag{A19}$$

which we define as $-i\lambda_1$ and $-i\lambda_2$, $-i\lambda_2$ being below the real axis.

Other branch points are at the roots of

$$-2iy(\tfrac{1}{4}A^2+m'^2)+A_1^2A_2^2A_3^2-C(iy) \tag{A20}$$

and those of

$$-2iy(\tfrac{1}{4}A^2+m^2)+iyA_j^2-C(iy) \tag{A21}$$

for $j=1-3$.

Depending on the phase of $-iC$, the roots of (A20) could be at $-iA_1^2A_2^2$, $-iA_2^2A_3^2$, and/or $-iA_3^2A_1^2$. The roots of (A21) could be at $4im'^2A_j^2$ and/or at $-iA_i^2A_k^2$ where i, j, and k are all different.

To find the phase of $-iC$, recall that on the real y axis we chose $-iC$ to have a positive imaginary part. At large positive y, we get

$$-iC\cong(iy^3)^{1/2}=e^{i\pi/4}y^{3/2}\ .$$

Letting $y=r_ye^{i\theta}$ and decreasing θ from 0 to $-\pi/2$ gives

$$-iC[i(-ir_y)]\cong -ir_y{}^{3/2}\ .$$

Following a similar procedure but continuing from large negative y gives the same result. Since the branch points of $-iC$ are all on the imaginary y axis, the phase will not change as we move up the y axis until $y=-i\lambda_2$ is reached. Passing this on the left decreases the phase by i, passing it on the right increases the phase by i, so we get that $-iC$ is negative and real for y between zero and $-i\lambda_2$ on the left and positive and real for y between zero and $-i\lambda_2$ on the right (Fig. 2).

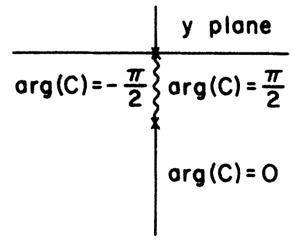

FIG. 2. The phase of $C(iy)$ on the negative imaginary y axis.

Both (A20) and (A21) can therefore only be zero for y below $-i\lambda_2$. For (A20), since the phase of $-iC$ is $-i$, to be zero the phase of the other part must be $+i$. This gives at $y=-iA_1^2A_2^2$ (for example)

$$A_3^2>A_1^2+A_1^2+4m^2\ . \tag{A22}$$

Notice that of the three possible branch points of (A20), only one can actually be a branch point on this sheet of C.

By similar reasoning for (A21), for $y=-iA_1^2A_2^2$ to be a branch point of $j=3$, we get the same condition as (A22). If this is a branch point of (A20), it will be one of (A21) and our integrand will vanish linearly. There is therefore only one branch point of the integrand of (A18) on the negative imaginary axis, namely, that at $-i\lambda_2$.

We wish to wrap the integration contour around the negative imaginary axis. At large y the integrand of (A18) goes as y^{-1}, but subtracting the term of zeroth order in the A's cancels this divergence. So we are free to distort the contour as stated above.

For y larger in magnitude than λ_2, the integrand is, with $y=i\tau$,

$$-i\sqrt{\tau}[2\tau(\tfrac{1}{4}A^2+m^2)+C(\tau)-A_1^2A_2^2A_3^2]^{3/2} \times\prod_{j=1}^{3}[2\tau(\tfrac{1}{4}A^2+m^2)+C(\tau)-\tau A_j^2]^{1/2}C(\tau)^{-1} \tag{A23}$$

multiplied by the phase $e^{-i\pi/4}$ on the right of the imaginary y axis and $-e^{-i\pi/4}$ on the left. For this part of the τ integration, we get

$$\frac{1}{8\pi^2}\int_{\lambda_2}^{\infty}d\tau\sqrt{\tau}[2\tau(\tfrac{1}{4}A^2+m^2)+C(\tau)-A_1^2A_2^2A_3^2]^{3/2}\prod_j[2\tau(\tfrac{1}{4}A^2+m^2)+C(\tau)-\tau A_j^2]^{-1/2}C(\tau)^{-1}\ . \tag{A24}$$

If we continue up the imaginary y axis past $-i\lambda_2$, C changes phase as described above. Except for the factor of C^{-1}, the integrand's phase changes by an infinitesimal amount as we move an infinitesimal distance past $-i\lambda_2$. This fixes the phase of the integrand in this region, giving for this piece

$$\begin{aligned}&\frac{1}{16\pi^2}\int_0^{\lambda_2}d\tau\sqrt{\tau}\Big\{-i[2\tau(\tfrac{1}{4}A^2+m^2)+i\,|C|-A_1^2A_2^2A_3^2]^{3/2}\prod_j[2\tau(\tfrac{1}{4}A^2+m^2)+i\,|C|-\tau A_j^2]^{-1/2}\\&\qquad +i[2\tau(\tfrac{1}{4}A^2+m^2)-i\,|C|-A_1^2A_2^2A_3^2]^{3/2}\prod_j[2\tau(\tfrac{1}{4}A^2+m^2)-i\,|C|-\tau A_j^2]^{-1/2}\Big\}|C|^{-1}\\&=\frac{1}{8\pi^2}\int_0^{\lambda_2}d\tau\sqrt{\tau}\,\mathrm{Re}\Big\{[2\tau(\tfrac{1}{4}A^2+m^2)+C-A_1^2A_2^2A_3^2]^{3/2}\prod_j[2\tau(\tfrac{1}{4}A^2+m^2)+C-\tau A_j^2]^{-1/2}C^{-1}\Big\}\ .\end{aligned} \tag{A25}$$

(A24) plus (A25) gives

$$\frac{\partial^2 f}{(\partial m^2)^2}=\frac{1}{8\pi^2}\int_0^\infty d\tau\sqrt{\tau}\,\mathrm{Re}\left\{[2\tau(\tfrac{1}{4}A^2+m^2)+C(\tau)-A_1{}^2A_2{}^2A_3{}^2]^{3/2}\prod_j[2\tau(\tfrac{1}{4}A^2+m^2)+C(\tau)-\tau A_j{}^2]^{-1/2}C(\tau)^{-1}\right\} . \tag{A26}$$

The phase of $C(\tau)$ is again, $+i$ for $\tau<\lambda_2$ and $+1$ for $\tau>\lambda_2$.

We still need to regulate the divergent parts. For the τ integration, this can be done by subtracting the term of zeroth order in A which is

$$\frac{1}{8\pi^2}\int_0^\infty d\tau\frac{1}{\sqrt{\tau}(4m^4+\tau)^{1/2}} . \tag{A27}$$

The m'^2 integration is still divergent. The units of $\partial^2 f/(\partial m^2)^2$ are m^0, so the term of nth order in the A's must be proportional to m^{-n}. Thus all terms of order greater than four will be convergent, as mentioned before. Subtracting the second- and fourth-order terms gives a finite result.

The second-order term is

$$-\frac{1}{8\pi^2}A^2\int_0^\infty\frac{d\tau}{\sqrt{\tau}(4m^4+\tau)^{1/2}}\left[\frac{1}{2[2m^2+(4m^4+\tau)^{1/2}]}-\frac{m^2}{4m^4+\tau}\right] . \tag{A28}$$

The τ integral gives zero for this.

The fourth-order term is

$$\begin{aligned}\frac{1}{8\pi^2}\int_0^\infty\frac{d\tau}{\sqrt{\tau}(4m^4+\tau)^{1/2}}\Bigg\{&(A^2)^2\Bigg[\frac{3}{8[2m^2+(4m^4+\tau)^{1/2}]^2}-\frac{1}{4(4m^4+\tau)^{1/2}[2m^2+(4m^4+\tau)^{1/2}]}-\frac{\tau}{8(4m^4+\tau)^2}\\&-\frac{m^2}{2(4m^4+\tau)[2m^2+(4m^2+\tau)^{1/2}]}+\frac{m^4}{(4m^4+\tau)^2}\Bigg]\\&+\sum_{i<j}A_i{}^2A_j{}^2\left[\frac{1}{2(4m^4+\tau)}-\frac{1}{2[2m^2+(4m^4+\tau)^{1/2}]^2}\right]\Bigg\}\end{aligned} \tag{A29}$$

whose τ integral gives $(1/48\pi^2m^4)\sum_{i<j}A_i{}^2A_j{}^2$. Notice that this regularization is gauge invariant insofar as (A28) vanishes, so there is no gluon mass term generated, and there is no $(A^2)^2$ term from (A29).

We can express z_1 (A16) as

$$z_1(\tau)=2\tau(\tfrac{1}{4}A^2+m^2)+C(\tau) .$$

A finite expression for f is

$$\begin{aligned}f=\frac{1}{8\pi^2}\int_{m^2}^\infty dm'^2(m'^2-m^2)\Bigg[&\int_0^\infty d\tau\sqrt{\tau}\,\mathrm{Re}\Bigg\{[z_1(\tau)-A_1{}^2A_2{}^2A_3{}^2]^{3/2}\prod_j[z_1(\tau)-\tau A_j{}^2]^{-1/2}C(\tau)^{-1}\\&-\frac{1}{\tau(4m'^4+\tau)^{1/2}}\Bigg\}-\frac{1}{6m'^4}\sum_{i<j}A_i{}^2A_j{}^2\Bigg] .\end{aligned} \tag{A30}$$

As it is, this expression is not suitable for use in, say, lattice calculations, since it is not manifestly gauge or Lorentz invariant. As mentioned in the text, however, we choose the three components of A to reproduce the path-ordered exponential

$$P\exp\left[i\oint A\cdot dx\right]$$

for certain infinitesimal contours. We do this as follows.

If there were a gauge in which the vector potential was constant, or at least had no linear term at the point x, choosing a Lorentz frame and gauge to give a vector potential with only three components as above gives

$$\mathrm{tr}F_{\mu\nu}F_{\mu\nu}=\sum_{i<j}A_i{}^2A_j{}^2 , \tag{A31a}$$

$$\mathrm{tr}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}=\frac{3i}{2}A_1{}^2A_2{}^2A_3{}^2 , \tag{A31b}$$

$$\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho}-2F_{\mu\lambda}F_{\nu\lambda}F_{\mu\rho}F_{\nu\rho})=A_1{}^2A_2{}^2A_3{}^2(A_1{}^2+A_2{}^2+A_3{}^2) . \tag{A31c}$$

Carrying out the product over j in (A30) gives, for this factor,

$$\left[z_1{}^3-\tau z_1{}^2A^2+\tau^2 z_1\sum_{i<j}A_i{}^2A_j{}^2-\tau^3A_1{}^2A_2{}^2A_3{}^2\right]^{-1/2} .$$

So the three quantities we need are given by (A31), with A^2 from (A31c):

$$A^2=\frac{3i}{2}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho}-2F_{\mu\lambda}F_{\nu\lambda}F_{\mu\rho}F_{\nu\rho})$$
$$\times(\mathrm{tr}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu})^{-1}\ .$$

Defining

$$M_{\mu\nu}=P\exp\left[i\oint_{C_{\mu\nu}}A\cdot dx\right]-1\ ,$$

where $C_{\mu\nu}$ is the square contour of side a shown in Fig. 1, we have the familiar results that

$$M_{\mu\nu}=ia^2F_{\mu\nu}+O(a^3)$$

and

$$\mathrm{tr}M_{\mu\nu}=-\tfrac{1}{2}a^4(F_{\mu\nu})^2+O(a^5)\ .$$

We can therefore write

$$G_2\equiv\lim_{a\to 0}-\frac{2}{a^4}\mathrm{tr}\sum_{\mu\nu}M_{\mu\nu}=\mathrm{tr}\sum_{\mu\nu}F_{\mu\nu}F_{\mu\nu}\ , \tag{A32a}$$

$$G_3\equiv\lim_{a\to 0}\frac{2}{3a^6}\mathrm{tr}\sum_{\mu\nu\lambda}M_{\mu\nu}M_{\nu\lambda}M_{\lambda\mu}$$
$$=-\frac{2i}{3}\mathrm{tr}\sum_{\mu\nu\lambda}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}\ , \tag{A32b}$$

$$G_4\equiv\lim_{a\to 0}-\frac{1}{a^8}\mathrm{tr}\sum_{\mu\nu\lambda\rho}(M_{\mu\nu}M_{\mu\nu}M_{\lambda\rho}M_{\lambda\rho}$$
$$-2M_{\mu\lambda}M_{\nu\lambda}M_{\mu\rho}M_{\nu\rho})$$
$$=\mathrm{tr}\sum_{\mu\nu\lambda\rho}(F_{\mu\nu}F_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho}-2F_{\mu\lambda}F_{\nu\lambda}F_{\mu\rho}F_{\nu\rho})\ . \tag{A32c}$$

Combining (A31) and (A32) allows us to write f in a gauge- and Lorentz-invariant way:

$$\sum_{i<j}A_i^2A_j^2=G_2\ ,$$
$$A_1^2A_2^2A_3^2=G_3\ ,$$
$$A^2=G_4/G_3\ .$$

We finally get

$$f=\frac{1}{8\pi^2}\int_{m^2}^{\infty}dm'^2(m'^2-m^2)\left[\int_0^\infty d\tau\sqrt{\tau}\,\mathrm{Re}\left\{\left[[z_1(\tau)-G_3]^3\left[z_1^3(\tau)-\tau z_1^2(\tau)\frac{G_4}{G_3}+\tau^2z_1(\tau)G_2-\tau^3G_3\right]^{-1}\right]^{1/2}\right.\right.$$
$$\left.\left.\times C(\tau)^{-1}-\frac{1}{\tau(4m'^4+\tau)^{1/2}}\right\}-\frac{1}{6m'^4}G_2\right] \tag{A33}$$

and we repeat the expressions for

$$C(\tau)=\left[\tau^3+4\tau^2\left[\frac{G_4}{4G_3}+m^2\right]^2-\tau^2G_2-4\tau m^2G_3\right]^{1/2} \tag{A34}$$

and

$$z_1(\tau)=2\tau\left[\frac{G_4}{4G_3}+m^2\right]+C(\tau)\ . \tag{A35}$$

[1]For a recent discussion of problems involving fermion integrations, see D. Weingarten, IBM Thomas J. Watson Research Center report (unpublished).

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[8]See *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. E. Stegun (U.S. GPO, Washington, D.C., 1964), especially Chap. 5.